Fermion bag solitons in the massive Gross–Neveu and massive Nambu–Jona–Lasinio models in 1+1 dimensions: inverse scattering analysis

2006 J. Phys. A: Math. Gen. 39 6341

(http://iopscience.iop.org/0305-4470/39/21/S26)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6341–6348 doi:10.1088/0305-4470/39/21/S26

Fermion bag solitons in the massive Gross–Neveu and massive Nambu–Jona–Lasinio models in 1+1 dimensions: inverse scattering analysis

Joshua Feinberg[1,2,3] **and Shlomi Hillel**[2]

[1] Physics Department, University of Haifa at Oranim, Tivon 36006, Israel
[2] Physics Department, Technion, Israel Institute of Technology, Haifa 32000, Israel

E-mail: joshua@physics.technion.ac.il

Received 6 November 2005, in final form 23 February 2006
Published 10 May 2006
Online at stacks.iop.org/JPhysA/39/6341

Abstract
Formation of fermion bag solitons is an important paradigm in the theory of the hadron structure. We report here on our non-perturbative analysis of this phenomenon in the (1+1)-dimensional massive Gross–Neveu model, in the large-N limit. Our main result is that the extremal static bag configurations are reflectionless, as in the massless Gross–Neveu model. Explicit formulae for the profiles and masses of these solitons are presented. We also present a particular type of self-consistent reflectionless solitons which arise in the massive Nambu–Jona–Lasinio models, in the large-N limit.

PACS numbers: 11.10.Lm, 11.10.Kk, 11.15.Pg, 71.27.+a

1. Introduction

An important dynamical mechanism, by which fundamental particles acquire masses, is through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region [1] or a shell [2] in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume and gradient energy associated with the condensate. This picture has interesting phenomenological consequences [1, 3].

This dynamical distortion of the homogeneous vacuum condensate configuration, namely, formation of fermion bag solitons, was demonstrated explicitly by Dashen, Hasslacher and Neveu (DHN) [4] many years ago, who studied semiclassical bound states in the (1+1)-dimensional Gross–Neveu (GN) model [5], using the inverse scattering method [6].

[3] Talk delivered by JF.

Following DHN, Shei [7] has applied the inverse scattering method to study solitons in the (1+1)-dimensional Nambu–Jona–Lasinio (NJL) model [8] in the large-N limit.

Fermion bags in the GN model were discussed in the literature several other times since the work of DHN, using alternative methods [9–11]. For a review on these and related matters (with an emphasis on the relativistic Hartree–Fock approximation) see [12]. For a more recent review of static fermion bags in the GN model (with an emphasis on reflectionless backgrounds and supersymmetric quantum mechanics) see [13]. The large-N semiclassical DHN spectrum of these fermion bags turns out to be essentially correct also for finite N, as analysis of the exact factorizable S-matrix of the GN model reveals [14].

A variational calculation of these effects in the (1+1)-dimensional massive generalization of the Gross–Neveu model, which we will refer to as MGN, was carried in [15] a few years ago, and more recently in [16]. Very recently, we studied static fermion bags in the MGN model [17], which we obtained using the inverse-scattering formalism, thus avoiding the need to choose a trial variational field configuration. Our main result in [17] is that the extremal static bag configurations are reflectionless, as in the massless Gross–Neveu model.

The GN and massless NJL models are completely integrable, which is why their S-matrices can be computed exactly. Thus, one may wonder whether DHN's and Shei's successful computation of the soliton spectrum in these models was made possible due to their integrablity. Our successful computation of the soliton spectrum in the MGN model indicates that the answer to that question is negative, as the MGN model is believed to be not completely integrable, unlike its massless counterpart. This raises hope that the soliton spectrum of more generic and non-integrable low-dimensional models may be computed explicitly.

Additional strong motivation to studying these models stems from condensed matter physics. The same features of the (1+1)-dimensional GN and NJL models, which make them so important in particle physics, namely, that they are exactly solvable models (at large- and finite-N) which exhibit asymptotic freedom, chiral symmetry breaking, bear a rich soliton spectrum and their S-matrices can be computed exactly, make these models very important also in condensed matter physics. Indeed, these models describe the physics of conducting polymers [18] and one-dimensional inhomogeneous superconductors [19]. They have been also applied in the description of other strongly correlated electronic systems [20]. For a recent review of the application of integrable QFT models to problems in condensed matter physics see [21].

In a series of papers [22], Thies *et al* have recently completed the computation of the phase diagram of the MGN model. For a comprehensive review of their work see [23]. Periodic inhomogeneous condensates—soliton crystals—play an important role in that analysis. (The soliton profiles presented here are obtained in the limit where the intersoliton spacing in that crystal becomes very large. This occurs in the low 'baryon' density limit.) A nice and useful feature of [23] is that it gathers in one place the particle and condensed matter physics applications of the GN and MGN models (and their generalizations), and demonstrates, once again, the importance of cross-fertilization between these two disciplines.

The rest of this paper is organized as follows: In the following section we briefly review the results of [17], leaving technical details out. Then, in section 3, we show that a subclass of the reflectionless solitons of [17] arise self-consistently in the (1+1)-dimensional massive NJL (MNJL) model. The latter extends the results of [7, 24] for the massless NJL model. Solitons in the MNJL model were also recently studied in [16], where a derivative expansion was carried out around a particular soliton background of the corresponding massless NJL model.

2. Solitons in the massive Gross–Neveu model

One way of writing the action for the MGN model is

$$S = \int \mathrm{d}^2x \left\{ \sum_{a=1}^{N} \bar{\psi}_a [\mathrm{i}\partial\!\!\!/ - \sigma]\psi_a - \frac{1}{2g^2}(\sigma^2 - 2M\sigma) \right\}, \tag{1}$$

where ψ_a $(a = 1, \ldots, N)$ are the massive Dirac fermions and σ is an auxiliary field. Integrating out σ results in an equivalent form of (1), with quartic fermion self-interactions.

An obvious symmetry of (1) with its N Dirac spinors is $U(N)$. Actually, (1) is symmetric under the larger group $O(2N)$ [4] (see also section 1 of [13]). The fact that the symmetry group of (1) is $O(2N)$ rather than $U(N)$ is related to the fact that it is invariant against charge conjugation, like the massless GN model. Consequently, the energy eigenvalues of the Dirac equation associated with (1), $[\mathrm{i}\partial\!\!\!/ - \sigma(x)]\psi = 0$, come in $\pm\omega$ pairs.

As usual, the theory (1) can be rewritten with the help of the scalar flavour singlet auxiliary field $\sigma(x)$. Also as usual, we take the large-N limit holding $\lambda \equiv Ng^2$ fixed. Integrating out the fermions, we obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2}\int \mathrm{d}^2x(\sigma^2 - 2M\sigma) - \mathrm{i}N\,\mathrm{Tr}\log(\mathrm{i}\partial\!\!\!/ - \sigma). \tag{2}$$

Noting that $\gamma_5(\mathrm{i}\partial\!\!\!/ - \sigma) = -(\mathrm{i}\partial\!\!\!/ + \sigma)\gamma_5$, we can rewrite the $\mathrm{Tr}\log(\mathrm{i}\partial\!\!\!/ - \sigma)$ as $\frac{1}{2}\mathrm{Tr}\log[-(\mathrm{i}\partial\!\!\!/ - \sigma)(\mathrm{i}\partial\!\!\!/ + \sigma)]$. In this paper we focus on static soliton configurations. If σ is time independent, the latter expression may be further simplified to $\frac{T}{2}\int\frac{\mathrm{d}\omega}{2\pi}[\mathrm{Tr}\log(h_+ - \omega^2) + \mathrm{Tr}\log(h_- - \omega^2)]$, where $h_\pm \equiv -\partial_x^2 + \sigma^2 \pm \sigma'$, and where T is an infra-red temporal regulator. As it turns out, the two Schrödinger operators $h_\pm$ are isospectral (see appendix A of [13] and section 2 of [11]) and thus we obtain

$$S[\sigma] = -\frac{1}{2g^2}\int \mathrm{d}^2x(\sigma^2 - 2M\sigma) - \mathrm{i}NT\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\mathrm{Tr}\log(h_- - \omega^2). \tag{3}$$

In contrast to the standard massless GN model, the MGN model studied here is not invariant under the Z_2 symmetry $\psi \to \gamma_5\psi, \sigma \to -\sigma$, and the physics is correspondingly quite different. As a result of the Z_2 degeneracy of its vacuum, the GN model contains a soliton (the so-called CCGZ kink [4, 9, 11, 13, 25]) in which the σ field takes on equal and opposite values at $x = \pm\infty$. The stability of this soliton is obviously guaranteed by topological considerations. With any non-zero M the vacuum value of σ is unique and the CCGZ kink becomes infinitely massive and disappears. If any soliton exists at all, its stability has to depend on the energetics of trapping fermions.

Let us briefly recall the computation of the unique vacuum of the MGN model. We shall follow [15]. For an earlier analysis of the MGN ground state (as well as its thermodynamics) see [26]. Setting σ to a constant, we obtain from (3) the renormalized effective potential (per flavour) $V(\sigma,\mu) = \frac{\sigma^2}{4\pi}\log\frac{\sigma^2}{e\mu^2} + \frac{1}{\lambda(\mu)}\left[\frac{\sigma^2}{2} - M(\mu)\sigma\right]$, where μ is a sliding renormalization scale with $\lambda(\mu) = Ng^2(\mu)$ and $M(\mu)$ is the running couplings. By equating the coefficient of σ^2 in two versions of V, one defined with μ_1 and the other with μ_2, we find immediately that $\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)} = \frac{1}{\pi}\log\frac{\mu_1}{\mu_2}$ and thus the coupling λ is asymptotically free, just as in the GN model. Furthermore, by equating the coefficient of σ in V we see that the ratio $\frac{M(\mu)}{\lambda(\mu)}$ is a renormalization group invariant. Thus, M and λ have the same scale dependence.

Without loss of generality we assume that $M(\mu) > 0$ and thus the absolute minimum of $V(\sigma,\mu)$, namely, the vacuum condensate $m = \langle\sigma\rangle$, is the unique (and positive) solution of the gap equation $\frac{\mathrm{d}V}{\mathrm{d}\sigma}\big|_{\sigma=m} = m\left[\frac{1}{\pi}\log\frac{m}{\mu} + \frac{1}{\lambda(\mu)}\right] - \frac{M(\mu)}{\lambda(\mu)} = 0$. Referring to (1), we see that m is the mass of the fermion. Using the explicit scale dependence of $\lambda(\mu)$, we can re-write the gap equation as $\frac{m}{\lambda(m)} = \frac{M(\mu)}{\lambda(\mu)}$, which shows manifestly that m, an observable physical quantity, is a renormalization group invariant. This equation also implies that $M(m) = m$, which makes sense physically.

Fermion bags correspond to inhomogeneous solutions of the saddle-point equation $\frac{\delta S}{\delta\sigma(x,t)} = 0$. In particular, static bags $\sigma(x)$ are the extremal configurations of the energy functional (per flavour) $\mathcal{E}[\sigma(x)] = -\frac{S[\sigma(x)]}{NT}$, subjected to the boundary condition that $\sigma(x)$ relaxes to its unique vacuum expectation value m at $x = \pm\infty$. More specifically, we have to evaluate the energy functional of a static configuration $\sigma(x)$, obeying the appropriate boundary conditions at spatial infinity, which supports K pairs of bound states of the Dirac equation at energies $\pm\omega_n, n = 1, \ldots, K$ (where, of course, $\omega_n^2 < m^2$). The bound states at $\pm\omega_n$ are to be considered together, due to the charge-conjugation-invariance of the GN model. Due to Pauli's exclusion principle, we can populate each of the bound states $\pm\omega_n$ with up to N fermions. In such a typical multiparticle state, the negative frequency state is populated by $N - h_n$ fermions (i.e., by h_n holes or antiparticles) and the positive frequency state contains p_n fermions (or particles). We shall refer to the total number of particles and antiparticles trapped in the nth pair of bound states $\nu_n = p_n + h_n$ as the valence, or occupation number of that pair.

The energy functional $\mathcal{E}[\sigma(x)]$ is, in principle, a complicated and generally unknown functional of $\sigma(x)$ and of its derivatives (which furthermore requires regularization). Thus, the extremum condition $\frac{\delta\mathcal{E}[\sigma]}{\delta\sigma(x)} = 0$, as a functional equation for $\sigma(x)$, seems intractable. The considerable complexity of the functional equations that determine the extremal $\sigma(x)$ configurations is the source of all difficulties that arise in any attempt to solve the model under consideration. DHN found a way around this difficulty in the case of the GN model [4]. They have used inverse scattering techniques [6] to express the (regulated) energy functional $\mathcal{E}[\sigma]$ in terms of the so-called scattering data associated with, e.g. the Hamiltonian h_- mentioned above (and thus with $\sigma(x)$), and then solved the extremum condition $\frac{\delta\mathcal{E}[\sigma]}{\delta\sigma(x)} = 0$ with respect to those data.

The scattering data associated with h_- are [6] the reflection amplitude $r(k)$ of the Schrödinger operator h_- at the momentum k, the number K of bound states in h_- and their corresponding energies $0 < \omega_n^2 \leqslant m^2, (n = 1, \ldots, K)$, and also additional K parameters $\{c_n\}$, where c_n has to do with the normalization of the nth bound state wavefunction ψ_n of h_-. More precisely, the nth bound state wavefunction, with energy ω_n^2, must decay as $\psi_n(x) \sim \text{const} \exp -\kappa_n x$ as $x \to \infty$, where $0 < \kappa_n = \sqrt{m^2 - \omega_n^2}$. If we impose that $\psi_n(x)$ be normalized, this will determine the constant coefficient as c_n. (With no loss of generality, we may take $c_n > 0$.) Recall that $r(-k) = r^*(k)$, since the Schrödinger potential $V(x) = \sigma^2(x) - \sigma'(x)$ is real. Thus, only the values of $r(k)$ for $k > 0$ enter the scattering data. The scattering data are independent variables, which determine $V(x)$ uniquely, assuming $V(x)$ belongs to a certain class of potentials which fall-off fast enough towards infinity. (Since the MGN does not bear topological solitons, neither h_- nor h_+ can have a normalizable zero energy eigenstate. Thus, all the ω_n are strictly positive.)

We can apply directly the results of DHN in order to write that part of $\mathcal{E}[\sigma(x)]$ which is common to the MGN and GN models, i.e. $\mathcal{E}[\sigma(x)]$ with its term proportional to M removed, in terms of the scattering data. For the lack of space we shall not write DHN's expression for the energy functional explicitly. Suffice it is to mention at this point that the 'DHN-part' of $\mathcal{E}[\sigma(x)]$ depends on the reflection amplitude only via certain regular dispersion integrals of the quantity $\log[1 - |r(k)|^2]$. The well-known reflectionless nature of solitons in the GN model is a direct consequence of this simple fact.

In order to complete the task of expressing the effective action of the MGN model in terms of the scattering data, we have to find such a representation for the remaining piece of $\mathcal{E}[\sigma(x)]$ proportional to M, namely, for $-\frac{M}{\lambda}\int_{-\infty}^{\infty}(\sigma(x)-m)\,\mathrm{d}x$. The latter integral cannot be expressed in terms of the scattering data based on the trace identities of the Schrödinger operator h_- discussed in appendix B of [4]. Evidently, new analysis is required to obtain its representation in terms of the scattering data. Happily enough, we were able to obtain such a representation in [17], which reads

$$\int_{-\infty}^{\infty}(\sigma(x)-m)\,\mathrm{d}x=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log[1-|r(k)|^2]}{\mathrm{i}m-k}\,\mathrm{d}k+\sum_{n=1}^{K}\log\left(\frac{m-\kappa_n}{m+\kappa_n}\right). \tag{4}$$

Thus, the M-dependent part of $\mathcal{E}[\sigma(x)]$, like its 'DHN-part', depends on the reflection amplitude only via the combination $\log[1-|r(k)|^2]$. Combining these two terms together, it follows that $\delta\mathcal{E}^{\mathrm{reg}}[\sigma(x)]/\delta r(k)=F(k)r^*(k)/(1-|r(k)|^2)$, where $F(k)$ is a calculable function, which does not vanish identically. Thus, $r(k)\equiv 0$ is the unique solution of the variational equation $\delta\mathcal{E}^{\mathrm{reg}}[\sigma(x)]/\delta r(k)=0$. Static extremal bags in the MGN model are *reflectionless*, as their counterparts in the GN model.

Explicit formulae for reflectionless $\sigma(x)$ configurations with an arbitrary number K of pairs of bound states are displayed in appendix B of [13]. In particular, the one which supports a single pair of bound states at energies $\pm\omega_b$ $\left(\kappa=\sqrt{m^2-\omega_b^2}\right)$, that originally discovered by DHN, is

$$\sigma(x)=m+\kappa\left[\tanh\left(\kappa x-\frac{1}{4}\log\frac{m+\kappa}{m-\kappa}\right)-\tanh\left(\kappa x+\frac{1}{4}\log\frac{m+\kappa}{m-\kappa}\right)\right]. \tag{5}$$

We see that the formidable problem of finding the extremal $\sigma(x)$ configurations of the energy functional $\mathcal{E}[\sigma]$ is reduced to the simpler problem of extremizing an ordinary function $\mathcal{E}(\omega_n,c_n)=\mathcal{E}[\sigma(x;\omega_n,c_n)]$ with respect to the $2K$ parameters $\{c_n,\omega_n\}$ that determine the reflectionless background $\sigma(x)$. If we solve this ordinary extremum problem, we will be able to calculate the mass of the fermion bag. This we did in detail in [17]. Let us sketch the procedure and state the final result.

The bare-regulated energy function $\mathcal{E}(\omega_n)$ which depends on the bare couplings λ and M and on the UV-cutoff Λ explicitly can be renormalized, in a manner essentially similar to the effective potential, as was described above. $\mathcal{E}$ is independent of c_n's, which appear in the scattering data. (The latter are thus flat directions for the energy function and determine the collective coordinates of the soliton.) The renormalized energy function thus obtained is a sum of the form $\sum_{n=1}^{K}f(\omega_n,\nu_n)$, where $f(\omega,\nu)$ is a known function, which depends also on the physical mass m explicitly, and also through the RG-invariant ratio $\gamma=\frac{1}{\lambda(m)}=\frac{M(\mu)}{m\lambda(\mu)}$. Thus, the extremum condition fixes each ω_n in terms of the number of the total number ν_n of particles and holes trapped in the bound states of the Dirac equation at $\pm\omega_n$, and not by the numbers of trapped particles and holes separately (see (6)). This fact is a manifestation of the underlying $O(2N)$ symmetry, which treats particles and holes symmetrically. Moreover, it indicates [17] that this pair of bound states gives rise to an $O(2N)$ antisymmetric tensor multiplet of rank ν_n of soliton states. (As it turns out, only tensors of ranks $0<\nu_n<N$ correspond to viable solitons [17].) The soliton as a whole is therefore the tensor product of all these antisymmetric tensor multiplets. Finally, we showed in [17] that only the irreducible ($K=1$) soliton was protected by energy conservation and $O(2N)$ symmetry against decaying into lighter solitons (or free massive fermions). Its profile is given by (5), where $\kappa=m\sin\theta$ (or, equivalently, $\omega=m\cos\theta$), with $0<\theta<\pi/2$, and where θ is determined by the extremum condition

$$\frac{\theta}{\pi}+\gamma\tan\theta=\frac{\nu}{2N}. \tag{6}$$

The left-hand side of (6) is a monotonically increasing function. Therefore, (6) has a unique solution in the interval $[0, \pi/2]$. This solution is evidently smaller than $\theta^{\mathrm{GN}} = \frac{\pi\nu}{2N}$, the corresponding value of θ in the GN model for the same occupation number. Thus, the corresponding bound state energy $\omega = m\cos\theta$ in the MGN model is higher than its GN counterpart, and thus less bound. The soliton mass (i.e., the renormalized energy function, evaluated at the solution of (6)) is

$$\mathcal{M}(\nu) = Nm\left(\frac{2}{\pi}\sin\theta + \gamma\log\frac{1+\sin\theta}{1-\sin\theta}\right). \tag{7}$$

This coincides with the corresponding results of variational calculations presented in [15, 16], which were based on (5) as a trial configuration. In fact, it was realized in [16] that the trial configuration (5) is an exact solution of the extremum condition $\frac{\delta\mathcal{E}[\sigma]}{\delta\sigma(x)} = 0$, provided (6) is used to fix $\kappa = m\sin\theta$.

3. Reflectionless solitons in the massive Nambu–Jona–Lasinio model

It is natural to enquire whether the results of the previous section carry over to the phenomenologically interesting MNJL model. The action for the MNJL model may be written as a generalization of (1), $S = \int \mathrm{d}^2x\left\{\sum_{a=1}^{N}\bar\psi_a[\mathrm{i}\partial\!\!\!/ - (\sigma + \mathrm{i}\pi\gamma_5)]\psi_a - \frac{1}{2g^2}(\sigma^2 + \pi^2 - 2M\sigma)\right\}$, where $\pi(x)$ is a pseudo-scalar auxiliary field. (Here we assumed that the 2×2 chiral mass matrix does not have a pseudo-scalar component, but this does not restrict the generality of our discussion in any way. This particular orientation of the mass matrix can be always reached at by performing a global—and therefore, anomaly free—chiral rotation in the σ–π plane.)

As in our discussion of the MGN model, we can integrate the fermions, and obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2}\int \mathrm{d}^2x(\sigma^2 + \pi^2 - 2M\sigma) - \mathrm{i}N\,\mathrm{Tr}\log(\mathrm{i}\partial\!\!\!/ - \sigma - \mathrm{i}\pi\gamma_5). \tag{8}$$

As before, we take the large-N limit, holding $\lambda \equiv Ng^2$ fixed. Unlike the NJL model, with its continuum of degenerate vacua, the ground state of the MNJL model (8) is unique, as in the MGN model. It corresponds to a constant field configuration, where $\pi = 0$ and where $\sigma = m$ is determined by an equation identical to that which arises in the MGN model.

Shei [7] has studied static solitons in the NJL model (i.e., $M = 0$ in (8)) using inverse scattering techniques. Similarly to DHN's results for the GN model, he concluded that extremal soliton profiles are reflectionless. Some of his results were rederived in [24], using a certain method based on properties of the diagonal resolvent of the Dirac operator (which was applied first to the GN model in [11]).

Could Shei's analysis be extended to study solitons in the MNJL model, similarly to the extension of DHN's inverse scattering analysis to the MGN model? Could it be that the self-consistent static soliton backgrounds in the MNJL model are reflectionless? It seems that all we need in order to answer these questions is a generalization of (4) to the case in which the Dirac operator involves a pseudo-scalar background $\pi(x)$. Unfortunately, we were not able (so-far) to find such a generalization, and therefore we cannot answer these questions in general at the moment. However, we were able to find a particular family of self-consistent reflectionless static solitons in the MNJL model by making an educated guess, as we shall now explain.

The spectrum of the Dirac equation associated with (8) is *not* invariant against charge conjugation, unless $\pi(x) \equiv 0$. Thus, the bound states corresponding to a static soliton background are not paired, in general. In particular, as has been shown by Shei, there exist solitons in the NJL model which bind fermions into a single bound state. However, he has also found solitons with the charge-conjugation-invariant spectrum (see equations (3.25)–(3.28) in [7]), with a pair of bound states $\pm\omega_b$, in which $\pi(x) = 0$ identically, and $\sigma(x)$ is given by (5), which thus coincide with the DHN solitons in the GN model, for which $\omega_b = m\cos\left(\frac{\pi\nu}{2N}\right)$ and $\mathcal{M}_{\text{DHN}}(\nu) = \frac{2Nm}{\pi}\sin\left(\frac{\pi\nu}{2N}\right)$. However, unlike in the GN model, in the NJL model we must choose $p = h = \frac{\nu}{2}$ (i.e., a soliton of this type must trap an equal number of fermions and anti-fermions). The reason for this restriction is not hard to understand physically: the total chiral rotation $\Delta\theta$, namely, the difference in $\arctan\frac{\pi(x)}{\sigma(x)}$ between the two ends of the one-dimensional space, must be related to the fermion number charge n_f deposited in the soliton according to $\Delta\theta = -\frac{2\pi n_f}{N}$ [27] (see also equations (5.10) and (5.22) in [24]). The soliton profile $(\sigma(x), \pi(x))$ under consideration starts at the vacuum point $(m, 0)$ at $x = -\infty$ and returns to it at $x = +\infty$. Thus, $\Delta\theta = n_f = p - h = 0$ for this soliton.

Now, any static soliton profile in the MNJL model must start at the *unique* vacuum $(m, 0)$ at $x = -\infty$ and return to it at $x = +\infty$. Thus, it should bring about null total chiral rotation, precisely as Shei's charge-conjugation-invariant configuration does. Therefore, if the MNJL bears *reflectionless* static solitons, they must be of this form (or charge-conjugate-invariant generalizations thereof, with more pairs of paired bound states). The only thing that should change compared to the NJL model is the quantization condition, relating ω_b and ν.

We have verified that this is indeed the case, simply by substituting this configuration into the static inhomogeneous saddle-point equations associated with (8). Varying (8) with respect to $\pi(x)$ we *obviously* obtain an equation identical in form to that of the NJL model. (For the latter, see the second equation in (5.1) in [24]). Using the explicit expressions for the entries of the diagonal resolvent of the Dirac operator with a reflectionless $(\sigma(x), \pi(x))$ background with two bound states (equations (4.13) and (4.14) in [24] with paired bound states $\omega_2 = -\omega_1$), we see that $\pi(x) \equiv 0$ is indeed a solution of that equation. (Here, having $p = h = \frac{\nu}{2}$ is essential.) This π-equation leaves ω_1 an undetermined function of ν. We still have to vary with respect to $\sigma(x)$. Substituting the explicit expressions for the appropriate entries of the diagonal resolvent of the Dirac operator (equations (4.13), (4.14) and (2.10) of [24]) in the saddle-point equation arising from variation with respect to $\sigma(x)$, and using the simplifying identity equation (2.24) of [28], we arrive simply at the static saddle-point equation of the MGN model, which is solved by $\sigma(x)$ given by (5) and the quantization condition (6), leading to soliton mass (7). Thus, a restricted subset of the extremal reflectionless solitons of the MGN model appear, not surprisingly, also in the MNJL model. For these solitons $\pi(x) \equiv 0$. The question whether these solitons exhaust all possibilities in the two-dimensional MNJL model remains open.

References

[1] Lee T D and Wick G 1974 *Phys. Rev.* D **9** 2291
Friedberg R, Lee T D and Sirlin R 1976 *Phys. Rev.* D **13** 2739
Friedberg R and Lee T D 1977 *Phys. Rev.* D **15** 1694
Friedberg R and Lee T D 1977 *Phys. Rev.* D **16** 1096
Chodos A, Jaffe R, Johnson K, Thorn C and Weisskopf V 1974 *Phys. Rev.* D **9** 3471

[2] Bardeen W A, Chanowitz M S, Drell S D, Weinstein M and Yan T M 1975 *Phys. Rev.* D **11** 1094
Creutz M 1974 *Phys. Rev.* D **10** 1749

[3] MacKenzie R, Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **53** 2203 (and references therein)

[4] Dashen R F, Hasslacher B and Neveu A 1975 *Phys. Rev.* D **12** 2443

[5] Gross D J and Neveu A 1974 *Phys. Rev.* D **10** 3235

[6] The literature on physical applications of inverse scattering theory is enormously vast. In the present paper we stick to the conventions of Novikov S, Manakov S V, Pitaevsky L P and Zakharov V E 1984 *Theory of Solitons—The Inverse Scattering Method* (New York: Consultants Bureau) (Contemporary Soviet Mathematics), which should be consulted for further references

[7] Shei S-S 1976 *Phys. Rev.* D **14** 535
[8] Nambu Y and Jona-Lasinio G 1961 *Phys. Rev.* **122** 345
Nambu Y and Jona-Lasinio G 1961 *Phys. Rev.* **124** 246
[9] See e.g., Klein A 1976 *Phys. Rev.* D **14** 558
Pausch R, Thies M and Dolman V L 1991 *Z. Phys.* A **338** 441
[10] Neveu A and Papanicolaou N 1978 *Commun. Math. Phys.* **58** 31
Papanicolaou N 1981 *Ann. Phys.* **136** 210
de Mello Koch R 1997 The large-N limit of fermionic systems *PhD Thesis* University of the Witwatersrand
[11] Feinberg J 1995 *Phys. Rev.* D **51** 4503
[12] Schön V and Thies M 2001 2D model field theories at finite temperature and density *At the Frontiers of Particle Physics—Handbook of QCD* vol 3 ed M Shifman (Singapore: World Scientific) (*Preprint* hep-th/0008175)
[13] Feinberg J 2004 *Ann. Phys.* **309** 166
[14] Zamolodchikov A B and Zamolodchikov Al B 1979 *Ann. Phys.* **120** 253
Karowski M and Thun H J 1981 *Nucl. Phys.* B **190** 61
For a recent discussion see Fendley P and Saleur H 2002 *Phys. Rev.* D **65** 025001
[15] Feinberg J and Zee A 1997 *Phys. Lett.* B **411** 134
[16] Thies M and Urlichs K 2005 *Phys. Rev.* D **71** 105008
[17] Feinberg J and Hillel S 2005 *Phys. Rev.* D **72** 105009 (*Preprint* hep-th/0509019)
[18] The literature on physics of conducting polymers is vast. A classic review is Heeger A J, Kivelson S, Schrieffer J R and Su W P 1988 *Rev. Mod. Phys.* **60** 781, and references therein
See also various papers in the special issue of *Synthetic Metals* 125 (2002), which celebrates the year 2000 Nobel Prize in chemistry
[19] See refs. 59, 61 and 62 in [23]: Mertsching J and Fischbeck H J 1981 *Phys. State Solid.* B **103** 783
Machida K and Nakanishi H 1984 *Phys. Rev.* B **30** 122
Buzdin A I and Kachkachi H 1997 *Phys. Lett.* A **225** 341
[20] Lin H H, Balents L and Fisher M P A 1998 *Phys. Rev.* B **58** 1794
[21] Essler F H L and Konik R M 2005 Applications of massive integrable quantum field theories to problems in condensed matter physics *From Fields to Strings: Circumnavigating Theoretical Physics* (The Ian Kogan Memorial Collection) vol 1, ed M Shifman, A Vainshtein and J Wheater (Singapore: World Scientific) pp 684–830 (*Preprint* cond-mat/0412421)
[22] The most recent papers in this series are Thies M and Urlichs K 2005 *Phys. Rev.* D **72** 105008
Schnetz O, Thies M and Urlichs K 2005 *Preprints* hep-th/0507120, hep-th/0511206
[23] Thies M *From Relativistic Quantum Fields to Condensed Matter and Back Again: Updating the Gross–Neveu Phase Diagram* an invited contribution to the Special Issue of the *J. Phys.* A on *Quantum Field Theory in Lower Dimensions and its Applications* (*Preprint* hep-th/0601049)
[24] Feinberg J and Zee A 1997 *Phys. Rev.* D **56** 5050
[25] Callan C G, Coleman S, Gross D J and Zee A unpublished
Gross D J 1976 *Methods in Field Theory* ed R Balian and J Zinn-Justin Les-Houches session XXVIII 1975 (Amsterdam: North Holland)
[26] Klimenko K G 1988 *Theor. Math. Phys.* **75** 487
Klimenko K G 1988 *Teor. Mat. Fiz.* **75** 226 (in Russian)
[27] Goldstone J and Wilczek F 1981 *Phys. Rev. Lett.* **47** 986
See also equation (4.44) of Aviv R and Zee A 1972 *Phys. Rev.* D **5** 2372
[28] Feinberg J 2002 *J. Math. Phys.* **43** 3927